SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

AUGUST 2005

Trial Higher School Certificate Examination

YEAR 12

Mathematics

General Instructions

- Reading time – 5 minutes.
- Working time – 3 Hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Start each **NEW** section in a separate answer booklet.

Total Marks - 120 Marks

- Attempt Questions 1 - 10
- All questions are of equal value.

Examiner: *E. Choy*

<u>**NOTE:**</u> This is a trial paper only and does not necessarily reflect the content or format of the final Higher School Certificate examination paper for this subject.

Total marks – 120
Attempt Questions 1 - 10
All questions are of equal value

Answer each SECTION in a SEPARATE writing booklet.

Section A

Marks

Question 1 (12 marks)

(a) If $x = 3\cdot517$ and $y = 1\cdot763$, find the value of **2**

$$\frac{x^2 + y^2}{xy}$$

correct to 3 significant figures

(b) Factorise $125a^3 - 1$ **2**

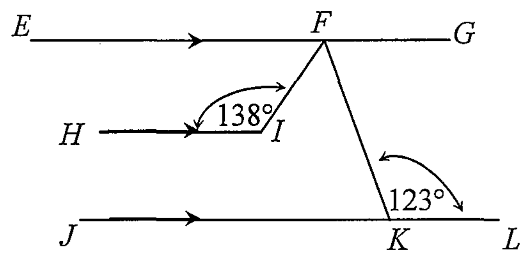
(c) Express **2**

$$\frac{5}{\sqrt{7} - 2}$$

with a rational denominator.

(d) Graph the solution set of $|2x + 1| < 3$ on a number line. **2**

(e) **2**



$EG \parallel HI \parallel JL$
$\angle HIF = 138°$ and $\angle FKL = 123°$

Find the size of $\angle IFK$ giving reasons.

(f) Find a primitive function of $8x - x^{-2}$ **2**

Question 2 (12 marks)

(a) $A(-7,0)$, $B(-9,3)$ and $C(0,9)$ are three points on the number plane.

(i) Show that AB and BC are perpendicular. **1**

(ii) Show that the line AB has equation $3x+2y+21=0$. **1**

(iii) Show that the length of AB is $\sqrt{13}$ units. **1**

(iv) Find the (exact) area of ΔABC **1**

(v) Find the (exact) perpendicular distance from O to BC. **2**

(vi) Write down **three** inequalities that define the region inside ΔABC. **3**

(b) (i) Sketch a graph of $y=|2x-6|$. **1**

(ii) Show graphically that the equation $|2x-6|=x$ has two solutions and find them. **1**

(iii) With the aid of your graphs, solve $|2x-6|<x$ **1**

End of Section A

Section B (Use a SEPARATE writing booklet)

Question 3 (12 marks) **Marks**

(a) Differentiate:

(i) $\left(7-3x^2\right)^4$ **1**

(ii) $6\ln x$ **1**

(iii) x^2e^{-x} **2**

(b) Find

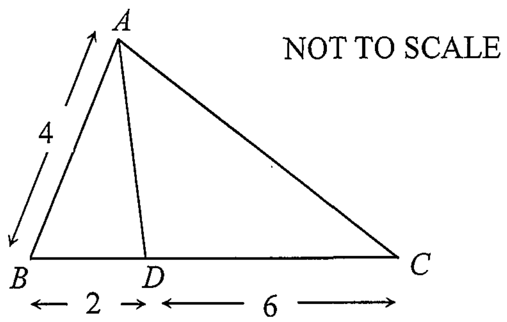
(i) $\int e^{3x}\,dx$ **1**

(ii) $\int 5\cos\left(\frac{x}{2}\right)dx$ **1**

(c) Evaluate $\int_1^e \frac{dx}{2x}$ **1**

(d) If $\frac{dy}{dx}=6x-9$ and $y=0$ when $x=1$, express y in terms of x. **2**

(e)



(i) In the diagram above prove that $\angle BDA=\angle BAC$. **1**

(ii) P, Q are the midpoints of sides AB and AC respectively of the triangle ABC. **2**
PQ is produced to R so that $PQ = QR$.

Prove that $CR=\frac{1}{2}AB$

Question 4 (12 marks)

(a) Let α and β be the roots of the equation $2x^2 - 5x + 1 = 0$.
Find the values of

(i) $\dfrac{5}{\alpha} + \dfrac{5}{\beta}$ **1**

(ii) $(\alpha - \beta)^2$ **1**

(b) Find the values of k for which the equation **2**

$$x^2 - (k-2)x + (k+1) = 0$$

has real roots.

(c) Find the equation of the normal to $y = 2x^2 - 3x + 1$ at the point $(-1, 6)$. **2**

(d) (i) By considering a suitable infinite geometric series, express $0\cdot\dot{4}$ as a fraction in simplest form. **1**

(ii) Express $\sqrt{0\cdot\dot{4}}$ in simplest precise decimal form.. **1**

(e) Find the coordinates of the centre and radius of the circle with equation **2**

$$x^2 + y^2 - 8x + y + \frac{1}{4} = 0$$

(f) Solve **2**

$$\log_2 x - \log_2 (x-2) = \frac{2}{3}\log_2 27$$

End of Section B

Section C (Use a SEPARATE writing booklet)

Question 5 (12 marks) **Marks**

(a) Consider the curve $y = x^3 + x^2 - x + 1$

(i) Find the coordinates of any turning points and determine their nature. **1**

(ii) Find any points of inflexion. **1**

(iii) Sketch the curve for $-2 \le x \le 2$. **1**

(iv) For what values of x is the curve concave up? **1**

(b) Find all the values of x with $0° \le x \le 360°$ for which **2**

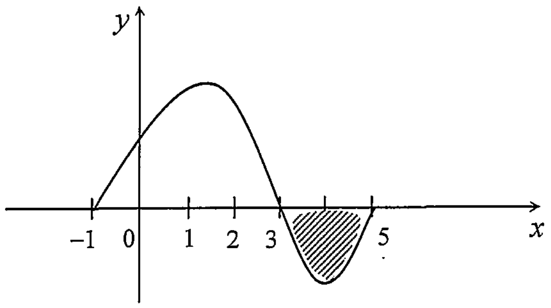
$$4\sin^2 x - 3 = 0$$

(c) A continuous curve $y = f(x)$ has the following properties for the closed interval $a \le x \le b$ **3**

$$f(x) > 0, f'(x) > 0, f''(x) < 0$$

Sketch a curve satisfying these conditions.

(d)



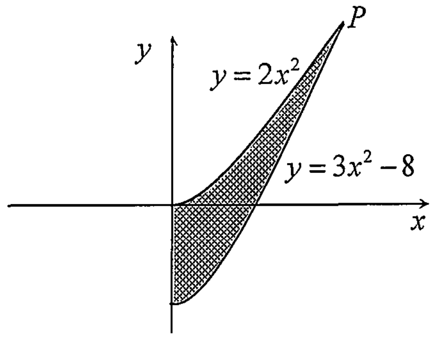
Given that $\int_{-1}^{3} f(x)\,dx = 11$ and that $\int_{-1}^{5} f(x)\,dx = 9$.

(i) What is the area of the shaded region? **1**

(ii) What is the value of $\int_{3}^{5} f(x)\,dx$? **2**

Question 6 (12 marks) **Marks**

(a)



P is a point of intersection of $y=2x^2$ and $y=3x^2-8$.

(i) Find the coordinates of P. **1**

(ii) Find the area of the shaded region. **2**

(b) Prove **2**

$$\frac{\sin\theta}{1-\cos\theta}=\frac{1+\cos\theta}{\sin\theta}$$

(c) Consider the geometric series

$$\sin^2\theta+\sin^2\theta\cos^2\theta+\sin^2\theta\cos^4\theta+\cdots$$

where $0<\theta<\dfrac{\pi}{2}$.

(i) Show that the sum, S_n, of the first n terms is given by **2**

$$S_n=1-\cos^{2n}\theta$$

(ii) Explain why this series always has a limiting sum. **1**

(iii) Let S be the limiting sum. **2**

Show that $S-S_n=\cos^{2n}\theta$

(iv) If $\theta=\dfrac{\pi}{3}$, find the least value of n for which $S-S_n<10^{-6}$ **2**

End of Section C

Section D (Use a SEPARATE writing booklet)

Question 7 (12 marks) **Marks**

(a) Twenty tickets were sold in a raffle. There are two prizes. First prize is two mobile phones. Second prize is one mobile phone.
You have bought two tickets.

What is the probability that you win

(i) three mobile phones? **1**

(ii) two mobile phones? **1**

(iii) no mobile phones? **1**

(iv) at least one mobile phone? **1**

(b) (i) Solve $2\sin x = \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ **2**

(ii) Show that $\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} = \ln 2$ **2**

(iii) Sketch on the same number plane, graphs of **2**

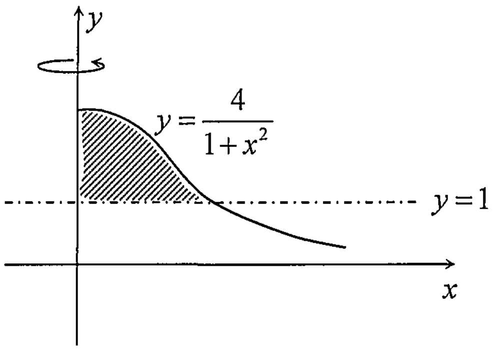
$$y = 2\sin x \text{ and } y = \tan x \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2},$$

showing the exact coordinates of their point of intersection.

(iv) Calculate the total area of the two closed regions formed by these graphs drawn in part (iii) above. **2**

Question 8 (12 marks) **Marks**

(a) **2**



The shaded region makes a revolution about the y axis.
Show that the volume of the resulting solid is given by

$$V = \pi \int_1^4 \left(\frac{4}{y} - 1 \right) dy$$

and find its *exact* volume.

(b) (i) Sketch the curve $y = \ln(x+1)$ for $-1 < x \leq 3$. **1**

(ii) The volume of the solid of revolution formed when the section of the curve $y = \ln(x+1)$ from $x = 0$ to $x = 2$ is rotated about the x axis is given by **3**

$$V = \pi \int_0^2 [\ln(x+1)]^2 \, dx$$

Use Simpson's Rule with five function values to approximate this integral.
(Leave your answer correct to three decimal places)

(c) The point $(1,1)$ is a stationary point on the curve $y = f(x)$. **2**
Find the equation of the curve given that $f''(x) = 2x - 3$.

(d) Suppose that $y = e^{kx}$.

(i) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. **2**

(ii) Find the value of k such that $y = 2\dfrac{dy}{dx} - \dfrac{d^2y}{dx^2}$. **2**

End of Section D

Section E (Use a SEPARATE writing booklet)

Question 9 (12 marks) **Marks**

(a) Let $f(x) = x^2 - \ln(2x-1)$

(i) Show that the domain of $f(x)$ is $x > \frac{1}{2}$. 1

(ii) Find $f'(x)$ and $f''(x)$. 2

(iii) Show that the x coordinates of any stationary points satisfy the equation $2x^2 - x - 1 = 0$. 1

(iv) Hence find the coordinates of any stationary points in the domain and by determining their nature, find the *minimum* value of the function. 2

(b) The Honda car company offers a loan of \$50 000 on CRV Sports 4 Wheel Drive purchased before 31st August 2005. The loan attracts an interest of just $0 \cdot 5\%$ per month.
To celebrate Honda's 25 years in Australia the company also offers an interest free period for the first six months. However, the first repayment is due at the end of the first month.
A customer takes out the loan and agrees to repay the loan over 10 years by making 120 equal monthly repayments of \$$M$.
Let A_n be the amount owing at the end of the nth repayment, then:

(i) Show that $A_6 = 50\,000 - 6M$ 1

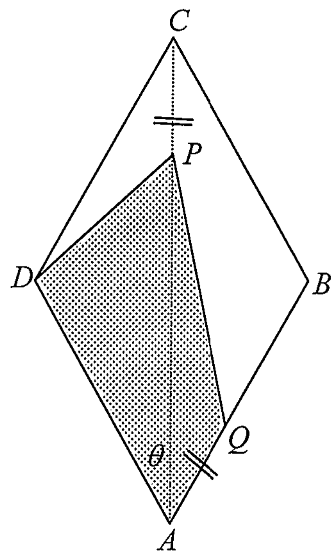
(ii) Show that $A_8 = (50\,000 - 6M) \times 1 \cdot 005^2 - M(1 \cdot 005 + 1)$ 2

(iii) Hence show that
$A_{120} = (50\,000 - 6M) \times 1 \cdot 005^{114} - 200M(1 \cdot 005^{114} - 1)$ 2

(iv) Find the value of the monthly repayment to the nearest cent. 1

Question 10 (12 marks) **Marks**

$ABCD$ is a rhombus of side 2 cm.
P and Q are points on AC and AB respectively such that $CP = AQ = x$ cm. $\angle DAP = \theta$ $\left(\text{where } 0 < \theta < \frac{\pi}{2}\right)$ and θ is a *constant*. Let the area of the shaded area $PDAQ$ be S cm^2.



(i) Show that $S = \frac{\sin\theta}{2}(4\cos\theta - x)(2 + x)$ **3**

(ii) If $\frac{dS}{dx} = 0$, find x in terms of θ. **2**

(iii) Find $\frac{d^2S}{dx^2}$ in terms of θ. **1**

(iv) Suppose that $\theta = \frac{\pi}{6}$, show that S attains its maximum when $\frac{PC}{AC} = \frac{\sqrt{3}-1}{2\sqrt{3}}$. **2**

(v) Suppose that $\theta = \frac{\pi}{4}$. A student says that S attains its maximum again when $\frac{PC}{AC} = \frac{\sqrt{3}-1}{2\sqrt{3}}$. **2**
Explain whether the student is correct.

(vi) Another student says that when P moves from C to A, where $0 < \theta < \frac{\pi}{2}$, S will increase to a certain maximum value and then decrease to 0. **2**
Explain whether this student is correct.

End of paper

SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

AUGUST 2005

Trial Higher School Certificate Examination

YEAR 12

Mathematics

Sample Solutions

Section	Marker
A	AF
B	DH
C	PB
D	CK
E	PP

Section A

1\. a. $\dfrac{(3.517)^2 + (1.763)^2}{(3.517)(1.763)}$

$= 2.50$

b. $(5a-1)(25a^2+5a+1)$

c. $\dfrac{5}{\sqrt{7}-2} \times \dfrac{\sqrt{7}+2}{\sqrt{7}+2}$

$= \dfrac{5\sqrt{7}+10}{7-4}$

$= \dfrac{5\sqrt{7}+10}{3}$

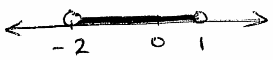
d. $|2x+1| < 3$

$2x+1<3$ or $-(2x+1)<3$

$2x<2$ $\quad -2x-1<3$

$x<1$ $\quad -2x<4$

$x>-2$



e. $\angle EFI = 42^\circ$ (co-interior angles are supplementary $EG // HI$)

$\angle GFK = 57^\circ$ (co-interior angles are supplementary $EG // JL$)

$\angle IFK = 81^\circ$ (angles on a line are supplementary)

f. $\int (8x - x^{-2})\,dx$

$= \dfrac{8x^2}{2} - \dfrac{x^{-1}}{-1} + C$

$= 4x^2 + \dfrac{1}{x}$

2\. a. i. $m = \dfrac{y_2-y_1}{x_2-x_1}$

$m_{AB} = \dfrac{3-0}{-9-(-7)}$

$= -\dfrac{3}{2}$

$m_{BC} = \dfrac{9-3}{0-(-9)}$

$= \dfrac{2}{3}$

$m_{AB} \times m_{BC} = -\dfrac{3}{2} \times \dfrac{2}{3}$

$= -1$

$\therefore AB \perp BC$

ii. $y - y_1 = m(x - x_1)$

$y - 0 = -\dfrac{3}{2}(x+7)$

$2y = -3x - 21$

$3x + 2y + 21 = 0$

iii. $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$AB = \sqrt{(-9+7)^2 + (3-0)^2}$

$= \sqrt{4+9}$

$= \sqrt{13}$ units

iv. $BC = \sqrt{(0-(-9))^2 + (9-3)^2}$

$= \sqrt{81+36}$

$= \sqrt{117}$

Area $= \dfrac{1}{2} \times \sqrt{13} \times \sqrt{117}$

$= \dfrac{39}{2}$ units^2

v. equation of BC

$y = \dfrac{2}{3}x + 9$

$3y = 2x + 27$

$2x - 3y + 27 = 0$

$Pd = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

$= \dfrac{|2(0) + (-3)(0) + 27|}{\sqrt{2^2 + (-3)^2}}$

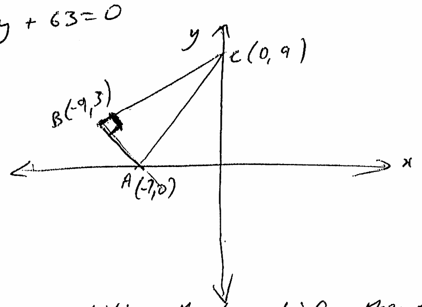
$= \dfrac{27}{\sqrt{13}}$ units.

vi. equation of AB is
$3x + 2y + 21 = 0$ Test (0,0) LHS = 21 ∴ $3x + 2y + 21 > 0$

equation of BC is
$2x - 3y + 27 = 0$ Test (0,0) LHS = 27 $2x - 3y + 27 > 0$

equation of AC is
$y = \frac{9}{7}x + 9$ Test (0,0) LHS = 63 $9x - 7y + 63 < 0$

$9x - 7y + 63 = 0$

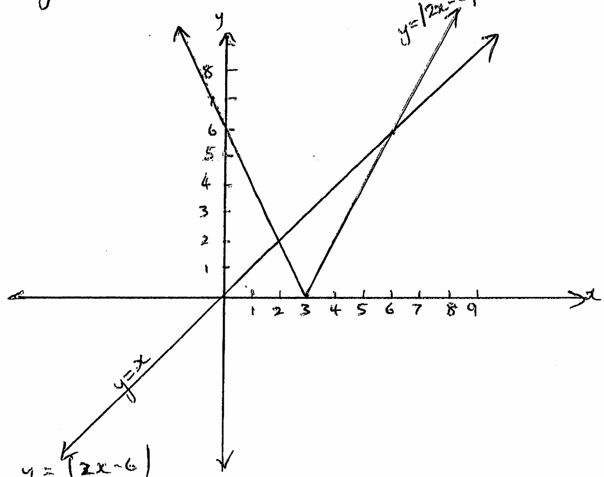


∴ the 3 inequalities that satisfy the region inside △ABC are $3x + 2y + 21 > 0 \cap 2x - 3y + 27 > 0 \cap 9x - 7y + 63 < 0$.

b. i. $y = |2x - 6|$

when $y = 0$, $2x - 6 = 0$
$2x = 6$
$x = 3$

when $x = 0$, $y = 6$



ii. you can see that $y = x$ and $y = |2x - 6|$ intersect at two points.
Therefore $|2x - 6| = x$ has two solutions. when $x = 2$ and when $x = 6$.

$x = 2x - 6$ or $x = -(2x - 6)$
$\underline{x = 6}$ $x = -2x + 6$
$3x = 6$
$\underline{x = 2}$

iii. where is the graph of $y = |2x - 6|$ less than the graph of $y = x$?
$\underline{2 < x < 6}$

Section B

3. (a) Differentiate:

i. $(7-3x^2)^4$

Solution: $4 \times (-6x)(7-3x^2)^3 = -24x(7-3x^2)^3$

ii. $6\ln x$

Solution: $\dfrac{6}{x}$

iii. x^2e^{-x}

Solution: $2xe^{-x} - x^2e^{-x}$

(b) Find

i. $\int e^{3x}\,dx$

Solution: $\dfrac{e^{3x}}{3} + c$

ii. $\displaystyle\int 5\cos\left(\frac{x}{2}\right)\,dx$

Solution: $2 \times 5\sin\dfrac{x}{2} + c = 1 - \sin\dfrac{x}{2} + c$

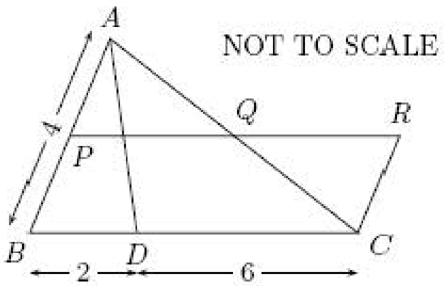
(c) Evaluate $\displaystyle\int_1^e \frac{dx}{2x}$

Solution: $\frac{1}{2}[\ln x]_1^e = \frac{1}{2}(1-0),$
$= \frac{1}{2}.$

(d) If $\dfrac{dy}{dx} = 6x - 9$ and $y = 0$ when $x = 1$, express y in terms of x.

Solution: $y = 3x^2 - 9x + c.$
$y = 0$ when $x = 1$, $\therefore\ 0 = 3 - 9 + c$, so $c = 6$.
Hence $y = 3x^2 - 9x + 6$.

(e)



i. In the diagram above prove that $\angle BDA = \angle BAC$.

Solution: $\angle ABD = \angle CBA$ (common)

$\frac{AB}{BC} = \frac{4}{8} = \frac{2}{4} = \frac{BD}{AB}$ (data)

$\therefore\ \triangle BAD\ ///\ \triangle BCA$ (2 sides same ratio, **i**ncluded angle equal)

$\therefore\ \angle BDA = \angle BAC$ (corresponding $\angle$s of similar $\triangle$s)

ii. P, Q are the midpoints of sides AB and AC respectively of the triangle ABC.
PQ is produced to R so that $PQ = QR$.

Prove that $CR = \frac{1}{2}AB$.

Solution: Method 1:

$$\begin{aligned} AQ &= QC \text{ (data)} \\ PQ &= QR \text{ (construction)} \\ \angle AQP &= \angle CQR \text{ (vertically opposite angles)} \\ \therefore\ \triangle APQ &\equiv \triangle CQR \text{ (SAS)} \\ AP &= CR \text{ (corresponding sides of congruent triangles)} \\ \text{but } AP &= \tfrac{1}{2}AB \text{ (} P \text{ bisects } AB\text{)} \\ \textit{i.e.},\ CR &= \tfrac{1}{2}AB. \end{aligned}$$

Solution: Method 2:

$$\left.\begin{aligned} PQ &\parallel BC \\ 2PQ &= BC \end{aligned}\right\} \text{(midpoint theorem for } \triangle\text{s)}$$

$2PQ = PR$ (construction)

$\therefore\ PBCR$ is a parm. (opposite sides equal and parallel)

Hence $RC = PB$ (opposite sides of parm.)

$\therefore\ CR = \frac{1}{2}AB$.

4. (a) Let α and β be the roots of the equation $2x^2 - 5x + 1$.
Find the values of

i. $\dfrac{5}{\alpha} + \dfrac{5}{\beta}$

Solution:

$$\begin{aligned} \alpha + \beta &= \frac{5}{2}, \\ \alpha\beta &= \frac{1}{2}. \\ \frac{5}{\alpha} + \frac{5}{\beta} &= 5\left(\frac{\alpha+\beta}{\alpha\beta}\right), \\ &= 5\left(\frac{\frac{5}{2}}{\frac{1}{2}}\right), \\ &= 25. \end{aligned}$$

ii. $(\alpha - \beta)^2$

Solution:

$$\begin{aligned} (\alpha - \beta)^2 &= \alpha^2 - 2\alpha\beta + \beta^2, \\ &= \alpha^2 + 2\alpha\beta + \beta^2 - 4\alpha\beta, \\ &= (\alpha + \beta)^2 - 4\alpha\beta, \\ &= \frac{25}{4} - \frac{4}{2}, \\ &= \frac{17}{4}. \end{aligned}$$

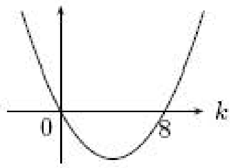
(b) Find the values of k for which the equation

$$x^2 - (k-2)x + (k+1) = 0$$

has real roots.

Solution:

$$\begin{aligned} \text{For real roots, } \Delta &\geq 0, \\ (k-2)^2 - 4(k+1) &\geq 0, \\ k^2 - 4k + 4 - 4k - 4 &\geq 0, \\ k^2 - 8k &\geq 0. \\ \therefore\ k \leq 0 \text{ or } k &\geq 8. \end{aligned}$$



(c) Find the equation of the normal to $y = 2x^2 - 3x + 1$ at the point $(-1, 6)$.

Solution:

$$\begin{aligned} \frac{dx}{dy} &= 4x - 3, \\ &= -7 \text{ when } x = -1. \\ \therefore\ y - 6 &= \tfrac{1}{7}(x+1), \\ 7y - 42 &= x + 1, \\ x - 7y + 43 &= 0. \end{aligned}$$

(d) i. By considering a suitable infinite geometric series, express $0{\cdot}\dot{4}$ as a fraction in simplest form.

Solution:
$$\begin{aligned} 0{\cdot}\dot{4} &= 0{\cdot}4 + 0{\cdot}4 \times 0{\cdot}1 + 0{\cdot}4 \times 0{\cdot}01 + 0{\cdot}4 \times 0{\cdot}001 + \ldots, \\ &= 0{\cdot}4(1 + 0{\cdot}1 + 0{\cdot}1^2 + 0{\cdot}1^3 + \ldots), \\ &= \frac{0{\cdot}4}{1 - 0{\cdot}1}, \\ &= \frac{4}{9}. \end{aligned}$$

ii. Express $\sqrt{0{\cdot}\dot{4}}$ in simplest precise decimal form.

Solution:
$$\begin{aligned} \sqrt{0{\cdot}\dot{4}} &= \sqrt{\tfrac{4}{9}}, \\ &= \tfrac{2}{3}, \\ &= 0{\cdot}\dot{6}. \end{aligned}$$

(e) Find the coordinates of the centre and the radius of the circle with equation

$$x^2 + y^2 - 8x + y + \frac{1}{4} = 0$$

Solution:
$$\begin{aligned} x^2 - 8x \qquad + y^2 + y \qquad &= -\tfrac{1}{4}, \\ x^2 - 8x + 16 + y^2 + y + \tfrac{1}{4} &= -\tfrac{1}{4} + 16 + \tfrac{1}{4}, \\ (x-4)^2 + (y + \tfrac{1}{2})^2 &= 4^2. \end{aligned}$$

$\therefore$ Centre $(4,\ -\tfrac{1}{2})$, radius 4.

Section C

<u>QUESTION 5</u> (a) $y = x^3 + x^2 - x + 1$.

$$\frac{dy}{dx} = 3x^2 + 2x - 1$$

$$\frac{d^2y}{dx^2} = 6x + 2.$$

(i) For turning points $\frac{dy}{dx} = 0$

$$3x^2 + 2x - 1 = 0.$$

$$(3x-1)(x+1) = 0.$$

$$x = -1, \tfrac{1}{3}.$$

$$\& \ y = 2, \tfrac{22}{27}$$

at $(-1, 2)$ $y'' = -4$ ∴ MAX. TURNING PT.

at $\left(\frac{1}{3}, \frac{22}{27}\right)$ $y'' = 4$ ∴ MIN. TURNING PT.

(ii) For points of inflexion, consider $\frac{d^2y}{dx^2} = 0$

$$6x + 2 = 0$$

$$6x = -2$$

$$x = -\tfrac{1}{3}$$

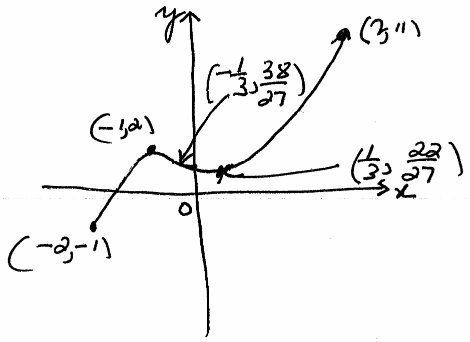
Testing

x	-1	$-\frac{1}{3}$	0
y''	-4	0	2

change in concavity

∴ $\left(-\frac{1}{3}, \frac{38}{27}\right)$ is a point of inflexion.

(iii)



(iv) For concave up

$$\frac{d^2y}{dx^2} > 0.$$

$$6x + 2 > 0$$

$$6x > -2$$

$$x > -\tfrac{1}{3}$$

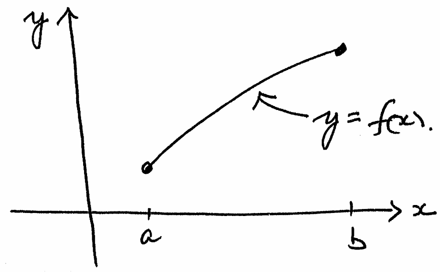
(b) $4\sin^2 x - 3 = 0$.

$\sin^2 x = \frac{3}{4}$

$\sin x = \pm\frac{\sqrt{3}}{2}$.

$x = 60°, 120°, 240°, 300°$.

(c)



positive, increasing and concave down.

(d) (i) $11 + A = 9$

$\therefore A = -2$.

$\therefore$ Area $= 2\ u^2$.

(ii)

$$\int_{-1}^{5} f(x)\,dx = \int_{-1}^{3} f(x)\,dx + \int_{3}^{5} f(x)\,dx$$

$$9 = 11 + \int_{3}^{5} f(x)\,dx.$$

$$\therefore \int_{3}^{5} f(x)\,dx = -2$$

QUESTION 6.

(a) (i) $3x^2 - 8 = 2x^2$

$x^2 = 8$

$x = 2\sqrt{2}$ (NB. $x > 0$)

$y = 16$

$\therefore$ P is $(2\sqrt{2}, 16)$

(ii) $A = \int_0^{2\sqrt{2}} \left[2x^2 - (3x^2 - 8)\right] dx.$

$= \int_0^{2\sqrt{2}} (8 - x^2)\, dx$

$= \left[8x - \frac{x^3}{3}\right]_0^{2\sqrt{2}}$

$= 16\sqrt{2} - \frac{16\sqrt{2}}{3}$

$= \frac{32\sqrt{2}}{3} \text{ u}^2$

(b) LHS $= \frac{\sin\theta}{1 - \cos\theta}$

$= \frac{\sin\theta}{1 - \cos\theta} \times \frac{1 + \cos\theta}{1 + \cos\theta}$

$= \frac{\sin\theta(1 + \cos\theta)}{1 - \cos^2\theta}$

$= \frac{\sin\theta(1 + \cos\theta)}{\sin^2\theta}$

$= \frac{1 + \cos\theta}{\sin\theta}$

$=$ RHS

(c) (i) $S_n = \dfrac{a(1-r^n)}{1-r}$

$= \dfrac{\sin^2\theta\left(1-(\cos^2\theta)^n\right)}{1-\cos^2\theta}$

$= \dfrac{\sin^2\theta\left(1-\cos^{2n}\theta\right)}{\sin^2\theta}$

$= 1-\cos^{2n}\theta$

(ii) $r = \cos^2\theta$ and $0 < \cos\theta < 1$ for $0 < \theta < \frac{\pi}{2}$

$\therefore$ $\boxed{0 < \cos^2\theta < 1}$

(NB. limiting sum exists where $|r| < 1$.)

(iii) $S = \dfrac{\sin^2\theta}{1-\cos^2\theta}$ (ie $\frac{a}{1-r}$)

$= \dfrac{\sin^2\theta}{\sin^2\theta}$

$= 1$

$\therefore S - S_n = 1-(1-\cos^{2n}\theta)$

$= \boxed{\cos^{2n}\theta.}$

(iv) If $\theta = \frac{\pi}{3}$, $(\cos^2\theta)^n = \left(\left(\frac{1}{2}\right)^2\right)^n$

$= 4^{-n}$

$\therefore$ If $S - S_n < 10^{-6}$ then $4^{-n} < 10^{-6}$

$4^n > 10^6$

$$\log_{10} 4^n > \log_{10} 10^6$$

$$n \log_{10} 4 > 6.$$

$$n > \frac{6}{\log_{10} 4}$$

$$> 9.965$$

$\therefore$ $\boxed{n = 10}$ is the least value.

Section D

Question 7

(a) W_1 = win 1st prize

W_2 = win 2nd prize

(i) $P(W_1 \text{ and } W_2) = \frac{2}{20} \cdot \frac{1}{19} = \frac{1}{190}$ 1

(ii) $P(W_1 \text{ and } \tilde{W}_2) = \frac{2}{20} \cdot \frac{18}{19} = \frac{9}{95}$ 1

(iii) $P(\tilde{W}_1 \text{ and } \tilde{W}_2) = \frac{18}{20} \cdot \frac{17}{19} = \frac{153}{190}$ 1

(iv) $1 - P(\text{no. phones}) = 1 - \frac{153}{190}$ 1

$= 37/190$

(b) (i) $2\sin x = \tan x$ 2

$2\sin x = \frac{\sin x}{\cos x}$

$\Rightarrow 2\sin x\cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$

$\therefore \sin x = 0$ or $\cos x = \frac{1}{2}$

$\boxed{x = 0}$ or $\boxed{\frac{\pi}{3}, -\frac{\pi}{3}}$

(ii) $\int_0^{\pi/3} \frac{\sin x}{\cos x}\,dx = \left[-\ln(\cos x)\right]_0^{\pi/3}$ 2

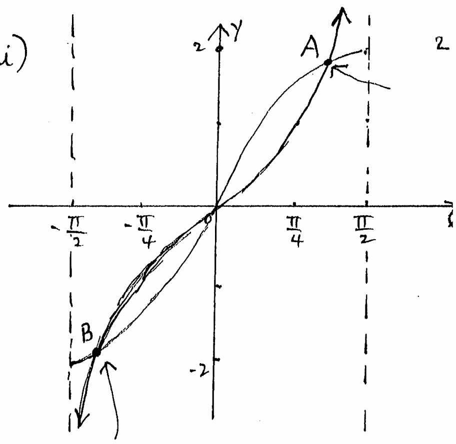
$= \left[\left(-\ln\cos\frac{\pi}{3}\right) - (-\ln\cos 0)\right]$

$= -\ln\frac{1}{2} + \ln 1$

$= -\ln 2^{-1} + 0$

$= \ln 2$

(iii)   2



$A\left(\frac{\pi}{3}, \sqrt{3}\right)$ $B\left(-\frac{\pi}{3}, -\sqrt{3}\right)$

(iv)

Area required   2

$= 2 \times \int_0^{\pi/3} (2\sin x - \tan x)\,dx$

$= 2 \times \left[\int_0^{\pi/3} 2\sin x\,dx - \int_0^{\pi/3} \tan x\,dx\right]$

$= 2 \times \left[-2\cos x\right]_0^{\pi/3} - 2 \times \ln 2$

$= 2 - 2\ln 2$ from (ii)

$= 2 - 2\ln 2$

Question 8

(a) Vol. about Y axis

$$V = \pi \int_c^d [g(y)]^2 \, dy$$

$$y = \frac{4}{1+x^2}$$

$$y + yx^2 = 4$$

$$x^2 = \frac{4-y}{y}$$

$$x^2 = \frac{4}{y} - 1$$

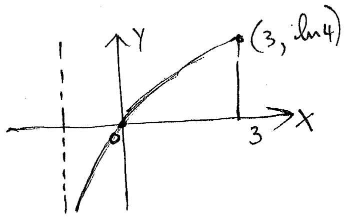
When $x=0$, $y=4$

$$\Rightarrow V = \pi \int_1^4 \left(\frac{4}{y} - 1\right) dy$$

$$= \pi \Big[4\log_e y - y \Big]_1^4$$

$$= \pi \Big[8\log_e 2 - 3 \Big]$$

(b) (i)



$$f(x) = [\ln(x+1)]^2$$

(ii)

$$\pi \int_0^2 [\ln(x+1)]^2 \, dx \doteqdot \pi \left\{ \frac{1}{6}\Big[f(0) + 4f(\tfrac{1}{2}) + f(1) \Big] + \frac{1}{6}\Big[f(1) + 4f(1\tfrac{1}{2}) + f(2) \Big] \right\}$$

$$\text{ie} \doteqdot \frac{\pi}{6}\Big[0 + 4(\ln \tfrac{3}{2})^2 + (\ln 2)^2 + (\ln 2)^2 + 4(\ln 2.5)^2 + (\ln 3)^2 \Big]$$

$$\doteqdot 3.238 \text{ units}^3$$

(c) $f(1) = 1$ and

$f'(1) = 0$

$$\Rightarrow f''(x) = 2x - 3$$

$$f'(x) = x^2 - 3x + c$$

$$\therefore f'(1) = -2 + c = 0 \quad \text{— (1)}$$

$$f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + cx + c_1$$

and $f(1) = \frac{1}{3} - \frac{3}{2} + 2 + c_1 = 1$ —(2

$$\therefore c_1 = \frac{1}{6}$$

$$\Rightarrow f(x) = \frac{x^3}{3} - \frac{3x^2}{2} + 2x + \frac{1}{6}$$

(d) (i) $y = e^{kx}$

$y' = ke^{kx}$,

$y'' = k^2 e^{kx}$

(ii) $e^{kx} = 2ke^{kx} - k^2 e^{kx}$

$1 = 2k - k^2$

$\Rightarrow k^2 - 2k + 1 = 0$

$(k-1)^2 = 0 \quad \therefore \boxed{k = 1}$

Section E

(9)(a) (i) $f(x) = x^2 - \ln(2x-1) \Rightarrow 2x-1>0 \quad [\because \ln u \text{ is defined for } u>0]$

$\therefore x > \frac{1}{2}$

(ii) $f'(x) = 2x - \frac{2}{2x-1}$

$= 2x - 2(2x-1)^{-1}$

$f''(x) = 2 + 2(2x-1)^{-2} \times 2$

$= 2 + \frac{4}{(2x-1)^2}$

(iii) Stationary points are when $f'(x) = 0$

$f'(x) = 2x - \frac{2}{2x-1} = 0$

$\therefore 2x(2x-1) - 2 = 0$

$\therefore x(2x-1) - 1 = 0$

$\therefore 2x^2 - x - 1 = 0$

(iv) $2x^2 - x - 1 = 0$

$\therefore (2x+1)(x-1) = 0$

$\therefore x = 1 \Rightarrow y = 1$

$f''(x) > 0$ for $x > \frac{1}{2}$, so $y = f(x)$ is **always** concave up.

So $(1,1)$ is the minimum point on the function.

So the minimum value is 1.

(b) (i) Even though it is interest free, the repayments are required each month.

$A_1 = 50\,000 - M$

$A_2 = A_1 - M = 50\,000 - 2M$

and so on for 6 months so that

$A_6 = 50\,000 - 6M$.

(ii) $A_7 = A_6(1\cdot005) - M$

$= (50\,000 - 6M)(1\cdot005) - M$

$A_8 = A_7(1\cdot005) - M$

$= [(50\,000 - 6M)(1\cdot005) - M](1\cdot005) - M$

$= (50\,000 - 6M)(1\cdot005)^2 - M(1 + 1\cdot005)$

(iii) For $n > 6$

$$A_n = (50\,000 - 6M)(1\cdot005)^{n-6} - M\left(1 + 1\cdot005 + \cdots + 1\cdot005^{(n-6)-1}\right)$$

$$= (50\,000 - 6M)(1\cdot005)^{n-6} - M\left(1 + 1\cdot005 + \cdots + 1\cdot005^{n-7}\right)$$

$$A_{120} = (50\,000 - 6M)(1\cdot005)^{114} - M\left(\underbrace{1 + 1\cdot005 + \cdots + 1\cdot005^{113}}_{114 \text{ terms}}\right)$$

$$= (50\,000 - 6M)(1\cdot005)^{114} - M \times \left(\frac{1\cdot005^{114} - 1}{1\cdot005 - 1}\right)$$

$$= (50\,000 - 6M)(1\cdot005)^{114} - M \times \left(\frac{1\cdot005^{114} - 1}{0\cdot005}\right)$$

$$= (50\,000 - 6M)(1\cdot005)^{114} - 200M\left(1\cdot005^{114} - 1\right)$$

(iv) $A_{120} = 0$

$$\therefore (50\,000 - 6M)(1\cdot005)^{114} - 200M\left(1\cdot005^{114} - 1\right) = 0$$

$$\therefore 50\,000(1\cdot005)^{114} - 206M(1\cdot005)^{114} + 200M = 0$$

$$\therefore M\left[206(1\cdot005)^{114} + 200\right] = 50\,000(1\cdot005)^{114}$$

$$\therefore M = \frac{50\,000(1\cdot005)^{114}}{206(1\cdot005)^{114} + 200} \approx \$539\cdot18$$

(10) (i) Let X be the intersection of the diagonals.

$\angle XAD = \angle XAC = \theta$ [property of rhombi]

$AX = 2\cos\theta \Rightarrow AC = 4\cos\theta$

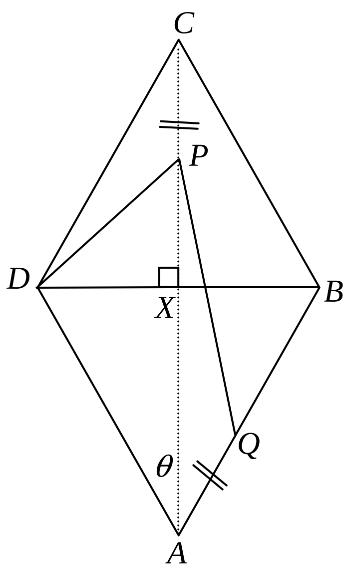
$\therefore AP = 4\cos\theta - x$

The shaded area is the sum of triangles ADP and APQ.

$$S = \frac{1}{2}\times 2\times\left(4\cos\theta - x\right)\sin\theta + \frac{1}{2}\times\left(4\cos\theta - x\right)\times x\sin\theta$$

$$= \frac{\sin\theta}{2}\left(4\cos\theta - x\right)\left(x+2\right)$$

[**NB** S is a concave down parabola in x]



(ii) $$S = \frac{\sin\theta}{2}\left[8\cos\theta + \left(4\cos\theta - 2\right)x - x^2\right]$$

$$\frac{dS}{dx} = \frac{\sin\theta}{2}\left[\left(4\cos\theta - 2\right) - 2x\right] = \sin\theta\left(2\cos\theta - 1 - x\right)$$

$$\therefore \frac{dS}{dx} = 0 \Rightarrow x = 2\cos\theta - 1 \qquad \left[\because \sin\theta \neq 0\right]$$

(iii) $$\frac{dS}{dx} = \sin\theta\left(2\cos\theta - 1 - x\right)$$

$$\therefore \frac{d^2S}{dx^2} = -\sin\theta \qquad \left[< 0 \text{ for } 0 < \theta < \frac{\pi}{2}\right]$$

(iv) $\theta = \frac{\pi}{6}$

$$\frac{dS}{dx} = 0 \Rightarrow x = 2\cos\left(\frac{\pi}{6}\right) - 1 = \sqrt{3} - 1$$

$$\frac{d^2S}{dx^2} < 0 \Rightarrow S \text{ is a maximum}$$

$$AC = 4\cos\left(\frac{\pi}{6}\right) = 2\sqrt{3}$$

$$\therefore \frac{PC}{AC} = \frac{\sqrt{3} - 1}{2\sqrt{3}}$$

(v) $\theta = \frac{\pi}{4}$

$$\frac{dS}{dx} = 0 \Rightarrow x = 2\cos\left(\frac{\pi}{4}\right) - 1 = \sqrt{2} - 1$$

$$\frac{d^2S}{dx^2} < 0 \Rightarrow S \text{ is a maximum}$$

$$AC = 4\cos\left(\frac{\pi}{4}\right) = 2\sqrt{2}$$

$$\therefore \frac{PC}{AC} = \frac{\sqrt{2} - 1}{2\sqrt{2}}$$

So the statement is **FALSE.**

(vi) If $\theta = \frac{\pi}{3}$ then

$$\frac{dS}{dx} = 0 \Rightarrow x = 2\cos\left(\frac{\pi}{3}\right) - 1 = 0$$

$$\frac{d^2S}{dx^2} < 0 \Rightarrow S \text{ is a maximum}$$

So if $\theta = \frac{\pi}{3}$ then S **STARTS** at its maximum value and then decreases to 0.

So the statement is **FALSE**.